# Measurements of non local weak values

**Aharon Brodutch**[1,2] **and Lev Vaidman**[1]

[1]Raymond and Beverly Sackler School of Physics and Astronomy, Tel Aviv University, Tel Aviv, Israel

[2]Center for Quantum Computer Technology, Faculty of Science, Macquarie University, Sydney NSW 2113, Australia

E-mail: `abrodutc@ics.mq.edu.au`

**Abstract.** Some recent attempts at measuring non local weak values via local measurements are discussed and shown to be less robust than standard weak measurements. A method for measuring some non local weak values via non local measurements (non local weak measurements) is introduced. The meaning of *non local weak values* is discussed.

## 1. INTRODUCTION

The concept of *observable* or *variable* in quantum mechanics is not as simple and intuitive as in classical physics and it requires definition. While in classical physics every observable has a value for any state of the system (which might be known or not known to the observer), in quantum mechanics we cannot associate a value for every observable. Instead, a measurement procedure is defined and every variable is associated with a set of *eigenvalues*, the possible outcomes of its measurement. Every state of a quantum system is associated with probability distributions for outcomes of measurements of every variable. It might happen that for some variables the probability distribution is singular, i.e., a particular eigenvalue is obtained with certainty and in this case, this eigenvalue can be associated with the variable as in the classical case. Otherwise, a statistical expectation value can be associated with a variable.

The definition of a quantum measurement procedure is, therefore, crucial for the concept of the value of a variable in quantum mechanics. The standard model of quantum measurement is the Von Neumann procedure [1] which consists of a short interaction between the system and a measuring device. The quantum measurement of a variable $O$ is described by the Hamiltonian :

$$H = g(t)PO, \tag{1}$$

where $P$ is the momentum conjugate to the pointer variable of the measuring device $Q$, and the normalized coupling function $g(t)$ specifies the time of the measurement interaction. The outcome of the measurement is the shift of the pointer variable during the interaction. In an ideal measurement the function $g(t)$ is nonzero only during a very short period of time, and the free Hamiltonian during this period of time can be neglected.

A discussion of ideal and non ideal measurements requires a few definitions. An *ideal, nondemilition, instantaneous*, measurement is defined as a Von Neumann measurement that leaves the state of a system, which initially was in an eigenstate of the measured observable, unchanged. A faithful demolition measurement is one that gives the required result but disturbs

the system by changing its state. Nonlocality too needs definition in this context. A nonlocal system is one that has two or more parts placed in separate locations. A local system is one that exists in a single location. Local interactions are those that can be created between two systems in the same place.

Formally, in non relativistic quantum mechanics one can consider any Hamiltonian and thus one can measure any variable. However, relativistic quantum mechanics limits us to local interactions and thus we cannot construct the Hamiltonian (1) for a variable $O$ related to a composite system with parts placed in separate locations. Nevertheless, measurements of some nonlocal variables are possible [2] in the sense that at the end of an instantaneous measurement procedure the composite system ends up in an eigenstate of this variable and the information about the eigenstate is written down (although in separate locations). We call this procedure a *nonlocal measurement.*

For local measurements we can consider the standard Von Neumann procedure with weakened coupling. In such measurements the pointer does not point sharply to zero before the interaction, and it does not point sharply to an eigenvalue after the interaction. The probability distribution of the pointer after the interaction points at the expectation value. Such measurements are called *weak measurements.* Particularly interesting are weak measurements performed on pre and post-selected quantum systems. For a quantum system pre-selected in a state $|\Psi\rangle$ and post-selected in a state $|\Phi\rangle$ the probability distribution of the pointer variable points to (a real part of) the *weak value* [3]

$$O_w \equiv \frac{\langle\Phi|O|\Psi\rangle}{\langle\Phi|\Psi\rangle}. \tag{2}$$

In some cases weak values might be much larger than the eigenvalues of $O$. This amplification is sometimes called the Aharonov-Albert-Vaidman (AAV) effect.

The simple formula above defines the weak value of any variable, including non local variables. In this paper we consider a possibility of combining the ideas of weak and nonlocal measurements to measure such non local weak values. This work was inspired by recently introduced concept of *joint weak measurements* [4]. In this procedure a new analysis of the readings on local measuring devices which performed local weak measurements allows to calculate the weak values of the product of two variables related to separate parts of the system. We analyze this procedure and show that it lacks some of the fundamental features of weak measurements and requires much larger resources than standard weak measurements. We show that some non local weak values can be measured using *non local weak measurements* and introduce a method for making such measurements. We argue that only values (local or non local) that can be measured directly using a weak measurement can be thought of as weak values, while those that are measured indirectly are just the result of a calculation leading to the result of (2).

## 2. Nonlocal Measurements

The special theory of relativity limits us to local interaction, so if $A$ is a nonlocal variable related to separate locations, the interaction described by (1) does not exist in nature. Nevertheless there are some nonlocal measurements which can be performed. A composite measuring device is used, with parts near every location of the variable $A$.

For example, a nonlocal variable $A = \sum B_i$ where each variable $B_i$ is related to location $i$ can be measured with the Hamiltonian

$$H = g(t) \sum P_i B_i, \tag{3}$$

where $P_i$ is the momentum conjugate to the pointer variables $Q_i$ located in location $i$.

If initially all pointer variables are well localized around zero, we can learn the outcome of a measurement of $A$ from the sum of readings of all local measuring devices $A = \sum Q_i$. This

procedure is a faithful measurement of $A$, but it is not an ideal measurement of $A$. In an ideal measurement a system initially in an eigenstate of $A$ should remain unchanged after the measurement. This is usually not the case for the method described above. For example, if our system consists of two spin$-\frac{1}{2}$ particles in a singlet state $\frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$ and $A = \sigma_{1z} + \sigma_{1z}$, then we will learn from our measurement that $A = Q_1 + Q_2 = 0$, but the singlet state will be changed either to $|\uparrow\rangle|\downarrow\rangle$ or to $|\downarrow\rangle|\uparrow\rangle$.

In order to perform the nonlocal measurement we should do the following [2]. Instead of localizing all pointers $Q_i$ we start with the following entangled state of the measuring device:

$$|\Psi\rangle_{in}^{MD} = |\sum Q_i = 0, P_1 = P_2 = ... = P_N\rangle \tag{4}$$

After the interaction, we can read the value of $A$ as before: $A = \sum Q_i$, but now the eigenstates of $A$ remain unchanged.

It has been shown [2] that beyond the measurement of a sum of local variables separated in space one can also perform a measurement of a modular sum of local variables. But there are nonlocal variables which cannot be measured in a non-demolition way [2]. In particular, consider a product, $\sigma_z^A \sigma_z^B$, of spin components of two separate spin-1 particles, one in Alice's hand and another in Bob's hand. We now prove that the possibility of an ideal non-demolition measurement of that variable contradicts causality, and thus, such a measurement does not exist.

Before time $t = 0$ we prepare the two particles in the initial state

$$|\Psi\rangle_{in} = \frac{1}{\sqrt{2}}(|-1\rangle_A + |0\rangle_A)\,|0\rangle_B \tag{5}$$

We assume that at time $t = 0$ somebody performs a non-demolition measurement of $\sigma_z^A \sigma_z^B$. Immediately after $t = 0$ Alice performs her projective local measurement of the state $\frac{1}{\sqrt{2}}(|-1\rangle_A + |0\rangle_A)$. Bob, who has access to particle $B$ can send a superluminal signal to Alice in the following way. Just before $t = 0$ he decides to change the state of his spin to $|1\rangle_B$ or to leave it as it is $|0\rangle_B$. If he decides to do nothing, then a nonlocal measurement of $\sigma_z^A \sigma_z^B$ will not change the state of the particles because state (5) is an eigenstate of $\sigma_z^A \sigma_z^B$. Therefore, Alice, in her local projective measurement will find the state $\frac{1}{\sqrt{2}}(|-1\rangle_A + |0\rangle_A)$ with certainty. However, if Bob decides to change the state of his spin to $|1\rangle_B$ the initial state before the nonlocal measurement at $t = 0$ will be changed to

$$|\Psi'\rangle_{in} = \frac{1}{\sqrt{2}}(|-1\rangle_A + |0\rangle_A)\,|1\rangle_B \tag{6}$$

It is not an eigenstate of $\sigma_z^A \sigma_z^B$ and thus after the measurement we will end up with equal probability with the state $|-1\rangle_A\,|1\rangle_B$ or $|0\rangle_A\,|1\rangle_B$. In both cases the probability of obtaining a positive outcome in Alice's's projective measurement on the state $\frac{1}{\sqrt{2}}(|-1\rangle_A + |0\rangle_A)$ is just one half. Instantaneous change of probability of a measurement performed by Alice breaks causality, therefore instantaneous measurement of $\sigma_z^A \sigma_z^B$ is impossible.

Note that not any product is unmeasurable. If instead of two spin-1 particles we consider two spin-$\frac{1}{2}$ particles then the product $\sigma_z^A \sigma_z^B$ is measurable. Indeed, we can express this product as a modular sum: $\sigma_z^A \sigma_z^B = (\sigma_z^A \sigma_z^B)mod4 - 1$ and every modular sum is measurable.

If we relax the requirement of an ideal measurement that it should be a non-demolition measurement, (which is a very rare property of real quantum measurements) and require only that it gives us a faithful result, then, conceptually, there are no constraints on measuring nonlocal variables. Given a large enough resource of entanglement we can "teleport" the quantum states of all separate parts of the system to one location and perform a measurement

of any variable [5]. This is not a real teleportation which requires sending classical bits, this procedure can be performed instantaneously. Of course, the result of measurement can only be read later, when the results of the local measurements will be brought together. If the system is in addition pre and post-selected, the procedure should be slightly modified, although (somewhat surprisingly) we need not add a lot of entanglement resources [6].

## 3. WEAK MEASUREMENTS

A weak measurement is a standard Von Neumann measurement with weakened interaction. One of the ways to weaken the interaction is to prepare the state of the measuring device in such a way that $P$ is very small. A good model of weak measurement is given by the coupling (1) with the initial state of the pointer variable a given by a Gaussian centered around zero:

$$\Psi_{in}^{MD}(Q) = (\Delta^2 \pi)^{-1/4} e^{-Q^2/4\Delta^2}. \tag{7}$$

with the position uncertainty $\Delta$ large ensuring a small $P$. We will henceforth use this model to describe weak measurements.

Weak values might lie very far from the range of the eigenvalues. For example, a spin half particle prepared in the $x$ direction and post-selected in an almost orthogonal state $\frac{1}{2}(\cos(\frac{\pi}{4}+\epsilon)|\uparrow\rangle - \sin(\frac{\pi}{4}+\epsilon)|\uparrow\rangle$ will yield a very large weak value for $\sigma_z$ [8]. Even in these cases there is an obvious shift of the pointer variable by the weak value (2). This shift gives weak values their significance [9].

While the meaning of weak values remains controversial [10, 11, 12], the justification of considering weak values as a description of the pre and post-selected quantum systems relies on the universality of the influence of the coupling to a variable in the limit of its weakness. The pointer variable prepared in a natural way (see Jozsa for some limitations [13]) shifts due to a weak measurement coupling as if it were coupled to a classical variable with the value equal to the weak value.

There have been numerous experiments showing weak values [14, 15, 16, 17, 18], mostly of photon polarization and the AAV effect has been well confirmed. Hosten and Kwiat [19] applied weak measurement procedure for measuring spin Hall effect in light. This effect is so tiny that it can not be observed without the amplification.

We must not forget that there is a method for performing any (demolition) measurement [5]. Thus, given large enough ensemble we can, in particular, measure the two-state vector at the time of the measurement. This is a complete description of a pre and post-selected quantum system, so it allows to calculate any function of the pre and post-selected state, and among others, the weak value or rather the result of (2).

## 4. LOCAL AND NONLOCAL MEASUREMENTS OF THE SUM OF LOCAL VARIABLES

We will start with the simplest example of non local variables, the sum of two local variables. Using the expression (2) we get

$$(A+B)_w = \frac{\langle \Phi | A + B | \Psi \rangle}{\langle \Phi | \Psi \rangle} \tag{8}$$

which in turn gives us the simple relation:

$$(A+B)_w = A_w + B_w \tag{9}$$

Thus, in order to find the value of $(A + B)_w$ we can measure $A_w$ and $B_w$ locally and just add the two numbers. This fact is somewhat surprising because the analogous relation for

an expectation values of the measuring devices performing strong measurements on pre and post-selected systems (which we signify as $\langle \quad \rangle_{\Phi\Psi}$) does not hold. In general

$$\langle A + B \rangle_{\Phi\Psi} \neq \langle A \rangle_{\Phi\Psi} + \langle B \rangle_{\Phi\Psi} \tag{10}$$

Consider the following example . At time $t_1$ two spin$-\frac{1}{2}$ particles are prepared in a state

$$|\Psi\rangle = \sqrt{\frac{1}{2+\epsilon^2}}(|\uparrow\rangle_A|\downarrow\rangle_B + |\downarrow\rangle_A|\uparrow\rangle_B) + \epsilon|\uparrow\rangle_A|\uparrow\rangle_B \tag{11}$$

Later, at time $t_2$, the particles are found in a state

$$|\Phi\rangle = \sqrt{\frac{1}{2+\epsilon^2}}(|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B) + \epsilon|\uparrow\rangle_A|\uparrow\rangle_B \tag{12}$$

We can use the Aharonov-Bergmann-Lebowitz (ABL)[20] formula for calculating the probabilities for the outcomes of strong intermediate measurements of a variable $C$ given the pre-selection $|\Psi\rangle$ and post-selection $|\Phi\rangle$

$$\text{Prob}(c_n) = \frac{|\langle\Phi|\mathbf{P}_{C=c_n}|\Psi\rangle|^2}{\sum_j |\langle\Phi|\mathbf{P}_{C=c_j}|\Psi\rangle|^2} \tag{13}$$

where $P_{c=c_j}$ is a projection operator for the eigenstate(s) with the eigenvalue $c_j$. Thus, for an ideal measurement of the nonlocal variable $\sigma_z^A + \sigma_z^B$ we obtain:

$$\langle\sigma_z^A + \sigma_z^B\rangle_{\Phi\Psi} = 2p(\uparrow\uparrow) + 0p(\sigma_z^A + \sigma_z^B = 0) - 2p(\downarrow\downarrow) = \tag{14}$$

$$\frac{2|\langle\Phi|\mathbf{P}_{\uparrow\uparrow}|\Psi\rangle|^2 - 2|\langle\Phi|\mathbf{P}_{\downarrow\downarrow}|\Psi\rangle|^2}{|\langle\Phi|\mathbf{P}_{\uparrow\uparrow}|\Psi\rangle|^2 + |\langle\Phi|\mathbf{P}_{\downarrow\downarrow}|\Psi\rangle|^2 + |\langle\Phi|\mathbf{P}_{\sigma_z^A+\sigma_z^B=0}|\Psi\rangle|^2} = \frac{2|\epsilon^2\frac{1}{2+\epsilon^2}|^2}{|\epsilon^2\frac{1}{2+\epsilon^2}|] + |0|^2 + |0|^2} = 2.$$

For an ideal measurement of a local variable $\sigma_z^A$, given that it is the only intermediate measurement that has been performed we have:

$$\langle\sigma_z^A\rangle_{\Phi\Psi} = p(\uparrow) - p(\downarrow) = \frac{|\langle\Phi|P_{\sigma_z^A=\uparrow}|\Psi\rangle|^2 - |\langle\Phi|P_{\sigma_z^A=\downarrow}|\Psi\rangle|^2}{|\langle\Phi|P_{\sigma_z^A=\uparrow}|\Psi\rangle|^2 + |\langle\Phi|P_{\sigma_z^A=\downarrow}|\Psi\rangle|^2} = \tag{15}$$

$$= \frac{|(\epsilon^2+1)\frac{1}{2+\epsilon^2}|^2 - |(-1)\frac{1}{2+\epsilon^2}|^2}{|(\epsilon^2+1)\frac{1}{2+\epsilon^2}|^2 + |(-1)\frac{1}{2+\epsilon^2}|^2} = \frac{2\epsilon^2 + \epsilon^4}{2 + \epsilon^4 + 2\epsilon^2}$$

The expectation value of $\sigma_z^B$ measured alone is

$$\langle\sigma_z^B\rangle_{\Phi\Psi} = p(\uparrow) - p(\downarrow) = \frac{|\langle\Phi|P_{\sigma_z^B=\uparrow}|\Psi\rangle|^2 - |\langle\Phi|P_{\sigma_z^B=\downarrow}|\Psi\rangle|^2}{|\langle\Phi|P_{\sigma_z^B=\uparrow}|\Psi\rangle|^2 + |\langle\Phi|P_{\sigma_z^B=\downarrow}|\Psi\rangle|^2} = \tag{16}$$

$$\frac{|(\epsilon^2-1)\frac{1}{2+\epsilon^2}|^2 - |(+1)\frac{1}{2+\epsilon^2}|^2}{|(\epsilon^2+1)\frac{1}{2+\epsilon^2}|^2 + |(+1)\frac{1}{2+\epsilon^2}|^2} = \frac{-2\epsilon^2 + \epsilon^4}{2 + \epsilon^4 - 2\epsilon^2}$$

It is easy to see that the expectation value of the nonlocal variable (14) is very different from the sum of (15) and (16). However, it is more reasonable to compare (14) with the sum of expectation values of the outcomes of local measurements of $\sigma_z^A$ and $\sigma_z^B$ performed

simultaneously. This corresponds to the measurement of a variable with non-degenerate eigenstates $|\uparrow\rangle_A|\downarrow\rangle_B, |\downarrow\rangle_A|\uparrow\rangle_B, |\uparrow\rangle_A|\uparrow\rangle_B, |\downarrow\rangle_A|\downarrow\rangle_B$. In this case we have

$$\langle\{\sigma_z^A\} + \{\sigma_z^B\}\rangle_{\Phi\Psi} = 2p(\uparrow\uparrow) + 0p(\uparrow\downarrow) + 0p(\downarrow\uparrow) = \frac{2|\langle\Phi|\mathbf{P}_{\uparrow\uparrow}|\Psi\rangle|^2}{|\langle\Phi|\mathbf{P}_{\uparrow\uparrow}|\Psi\rangle|^2+|\langle\Phi|\mathbf{P}_{\uparrow\downarrow}|\Psi\rangle|^2+|\langle\Phi|\mathbf{P}_{\downarrow\uparrow}|\Psi\rangle|^2} = \quad (17)$$

$$\frac{2|\epsilon^2\frac{1}{2+\epsilon^2}|^2}{|\epsilon^2\frac{1}{2+\epsilon^2}|^2 + |\frac{1}{2+\epsilon^2}|^2 + |(-1)\frac{1}{2+\epsilon^2}|^2} = \frac{2\epsilon^2}{2+\epsilon^4}$$

the brackets $\{\}$ signify a separate measurement of each observable.

We see that expectation value of the strong measurements of the sum of two variables related to separate parts is not equal to the sum of the expectation values of local measurements, even if they are performed simultaneously.

$$\langle\sigma_z^A + \sigma_z^B\rangle_{\Phi\Psi} \neq \langle\sigma_z^A\rangle_{\Phi\Psi} + \langle\sigma_z^B\rangle_{\Phi\Psi} \neq \langle\{\sigma_z^A\} + \{\sigma_z^B\}\rangle_{\Phi\Psi} \quad (18)$$

For weak measurements, however, the equality (9) holds, both for separate and joint local measurements. If we assume the existence of nonlocal interactions, then we can directly couple to the sum $\sigma_z^A + \sigma_z^B$.

$$H = g(t)P(\sigma_z^A + \sigma_z^B). \quad (19)$$

Since a strong measurement yields the outcome 2 with certainty, the weak value should also equal 2 according to the theorem proved in [21]. In this particular example, the coupling need not be weak to find the weak value 2. Since there are no nonlocal interactions in nature we have to apply local weak measurements of $\sigma_z^A$ and $\sigma_z^B$

$$H = g(t)(P_A\sigma_z^A + P_B\sigma_z^B) \quad (20)$$

and add the outcomes. We use a model in which the initial wave functions of the measuring devices are Gaussians around zero

$$\Psi_{in}^{MD}(Q_A,Q_B) = Ne^{\frac{-Q_A^2}{2\Delta^2}}e^{\frac{-Q_B^2}{2\Delta^2}} \quad (21)$$

The measurement interaction leads to a shift of the pointer wave function by the eigenvalue of $\sigma_z$ so that after the measurement, the measuring device will be described by the state

$$\Psi_{fin}^{MD}(Q_A,Q_B) =$$
$$\Psi_{in}^{MD}(Q_A - 1, Q_B + 1) - \Psi_{in}^{MD}(Q_A + 1, Q_B - 1)) + \epsilon^2\Psi_{in}^{MD}(Q_A - 1, Q_B - 1) = \quad (22)$$
$$Ne^{\frac{-(Q_A^2+Q_B^2)}{2\Delta^2}}[e^{\frac{-(-2Q_A+2Q_B+2)}{2\Delta^2}} - e^{\frac{-(2Q_A-2Q_B+2)}{2\Delta^2}} + \epsilon^2e^{\frac{-(-2Q_A-2Q_B+2)}{2\Delta^2}}]$$

with $N$ being a normalization constant.

The expectation value of the measurement outcome is:

$$\langle Q_A + Q_B\rangle = \int(Q_A + Q_B)|\Psi_{fin}^{MD}(Q_A,Q_B)|^2dQ_AdQ_B =$$
$$\frac{2\epsilon^4}{\epsilon^4+(2-2e^{\frac{-2}{\Delta^2}})} \approx 2 - \frac{8}{\Delta^2\epsilon^4} \quad (23)$$

for large $\Delta$. The statistical measurement error is the width $\Delta$. At the limit of weak measurements, i.e. large $\Delta$ it indeed yields the weak value $(\sigma_z^A + \sigma_z^B)_w = 2$. However, we can go close to the weak value only for very large $\Delta$, (very weak measurement) and thus very large uncertainty in the final reading. For example, in the case of $\epsilon = 0.1$ we will need to have

$\Delta > 600$ in order to measure the weak value with a deviation of 10%. Of course, if we make the same measurement on a large ensemble the statistical error will be made smaller according to $\Delta_n = \frac{\Delta}{\sqrt{n}}$ requiring us to use an ensemble of about $3.6 \times 10^5$ such pre and post selected systems just to be within the right order of magnitude. This number will be increased if we want to make the deviation or the statistical error smaller. Since for a true nonlocal measurement we can get the required result at the strong limit, we only need one such system to get the correct expectation value. We see that local measurements allow us to find nonlocal weak values, but it is a very inefficient procedure. The nonlocal weak value is the result of a calculation made on the readings of two pointer variables rather than the direct result of the reading of a single pointer variable.

Let us now try to combine the techniques of nonlocal measurements based on a measuring device with entangled parts and local interactions, with the weak measurement techniques. The measuring device in an ideal strong nonlocal measurement has an initial state (4). To make it weak we have to prepare the conjugate momenta to be centered around 0 which requires the pointer (centered around $Q_A + Q_B = 0$) to have a large uncertainty. In this case, the initial state of the measuring device (using the usual Gaussian model) will be

$$\Psi_{in}^{MD}(Q_A + Q_B) = Ne^{\frac{(Q_A+Q_B)^2}{2\Delta^2}} \tag{24}$$

and the final state

$$\Psi_{fin}^{MD}(Q_A + Q_B) = Ne^{\frac{(Q_A+Q_B-2)^2}{2\Delta^2}} \tag{25}$$

This is a Gaussian around the weak value $(\sigma_z^A + \sigma_z^B)_w = 2$. In fact for this example, the measurement need not be weak to get it right due to an accidental fact: the strong measurement also yields the eigenvalue "2" with certainty.

Let us consider another example to compare various measurement methods. The system is described by the two state vector

$$\langle \uparrow\downarrow + \downarrow\uparrow + \uparrow\uparrow + \downarrow\downarrow| \quad |0.95 \uparrow\downarrow -1.05 \downarrow\uparrow +0.11 \uparrow\uparrow\rangle \tag{26}$$

so that the weak value is

$$(\sigma_z^A + \sigma_z^B)_w = \frac{\langle \uparrow\downarrow + \downarrow\uparrow + \uparrow\uparrow + \downarrow\downarrow| \sigma_z^A + \sigma_z^B |-1.05 \uparrow\downarrow +0.95 \downarrow\uparrow +0.11 \uparrow\uparrow\rangle}{\langle \uparrow\downarrow + \downarrow\uparrow + \uparrow\uparrow | - 1.05 \uparrow\downarrow +0.95 \downarrow\uparrow +0.11 \uparrow\uparrow\rangle} = 22, \tag{27}$$

while the local weak values are

$$(\sigma_z^A)_w = 211 \tag{28}$$
$$(\sigma_z^B)_w = -189. \tag{29}$$

Using the measuring device with the initial state (24) we obtain the final state

$$\Psi_{fin}^{MD}(Q_A, Q_B) = N[-1.05e^{\frac{-(Q_A+Q_B)^2}{2\Delta^2}} + 0.95e^{\frac{-(Q_A+Q_B)^2}{2\Delta^2}} + 0.11e^{\frac{-(Q_A+Q_B-2)^2}{2\Delta^2}}], \tag{30}$$

and the expectation value of the pointer variable:

$$\langle Q_A + Q_B \rangle = \frac{22.0 \left(11.0e^{2.0/\Delta^2} - 10.0\right)}{221.0e^{2.0/\Delta^2} - 220.0} \approx 22 - \frac{2360}{\Delta^2}. \tag{31}$$

A deviation of 1% and an uncertainty of 10% will require an ensemble of about $2.2 \times 10^3$ particles see fig 1. If on the other hand we have only local weak measurements, i.e. two local measuring devices, we will get an expectation value of $\langle Q_A + Q_B \rangle \approx 22 - \frac{8.8 \times 10^5}{\Delta^2}$. Here a deviation of 1% and an uncertainty of 10 % will require an ensemble of about $8.2 \times 10^5$ particles. In this example we can see that entanglement in the measuring device provides an improvement of more than two orders of magnitude.
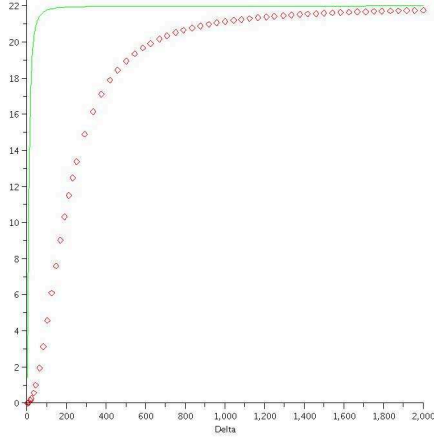
**Figure 1. Comparison of local and entangled measuring devices**. Expectation values of the pointer variable as a function of $\Delta$. Deviations of 10% are at values of $\Delta > 30$ for the nonlocal measuring device ( full line) and $\Delta > 600$ for the local one (dots) . While Deviations of 1% are at values of $\Delta > 100$ for the nonlocal measuring device and $\Delta > 2,000$ for the local one.

## 5. LOCAL AND NONLOCAL WEAK MEASUREMENTS OF THE PRODUCT OF LOCAL VARIABLES

We now have enough background to analyze recent results about the measurements of the product of separate local variables named "joint weak values" [4]. If we are given nonlocal (unphysical) interactions, then a weak measurement of the product is not different from any other weak measurement and all theory of local weak measurements is applicable. However, if we consider only local interactions, the situation is very different. The product rule does not hold, not only for expectation values of strong measurements

$$\langle AB \rangle_{\Phi\Psi} \neq \langle A \rangle_{\Phi\Psi} \langle B \rangle_{\Phi\Psi} \tag{32}$$

but also for weak values:

$$(AB)_w \neq A_w B_w. \tag{33}$$

At first glance it seems that local weak measurements cannot help us find the weak value of the product. Here is an example of two pre- and post-selected states which yield different values, of $(\sigma_z^A \sigma_z^B)_w$, but have the same *joint* probability for the pointer variables of local weak measurements. In both cases the initial state is the product $|\uparrow_x\rangle_A |\uparrow_x\rangle_B$, but the example is more transparent if we write it in spin $z$ basis

$$|\Psi\rangle = \frac{1}{2}(|\uparrow\rangle_A|\uparrow\rangle_B + |\downarrow\rangle_A|\downarrow\rangle_B + |\uparrow\rangle_A|\downarrow\rangle_B + |\downarrow\rangle_A|\uparrow\rangle_B) \tag{34}$$

The first post-selected stated state is

$$|\Phi\rangle = \frac{1}{2}[|\uparrow\rangle_A|\uparrow\rangle_B + |\downarrow\rangle_A|\downarrow\rangle_B + i(|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B)] \tag{35}$$

and the second is

$$|\Phi'\rangle = \frac{1}{2}[|\uparrow\rangle_A|\uparrow\rangle_B - |\downarrow\rangle_A|\downarrow\rangle_B + i(|\uparrow\rangle_A|\downarrow\rangle_B + |\downarrow\rangle_A|\uparrow\rangle_B)] \tag{36}$$

It is easy to see that in the first case $(\sigma_z^A \sigma_z^B)_w = 1$ while in the second case $(\sigma_z^A \sigma_z^B)_w = -1$. In fact, it is a special case in which the weak value is equal to the result which is obtained with certainty in a strong nonlocal measurement.

For local measurement, we use the same model as before and write down the state of the two local measuring devices as a product of Gaussians

$$\Psi_{in}^{MD}(Q_A, Q_B) = e^{-\frac{Q_A^2}{4\Delta^2}} e^{-\frac{Q_B^2}{4\Delta^2}} \tag{37}$$

After the interaction and post-selection, the joint distribution for the pointer variables $Q_a, Q_B$ given by $\Psi^\dagger \Psi$ turns out to be the same in both cases, so that a measurement of $Q_A, Q_B$ or any combination of the two will not provide us with a method for distinguishing between the two initial states.

Nevertheless, Resch and Steinberg showed that one can find the weak value of the product by looking at the local measuring device. They proved the following formula (for a measuring device initially centered around zero)

$$Re(AB)_w = 2\langle Q_A Q_B \rangle - Re(A_w^* B_w) \tag{38}$$

Resch and Laudeen provided another expression

$$Re(AB)_w = \langle Q_A Q_B \rangle - \frac{4\Delta^4}{\hbar^2}\langle P_A P_B \rangle \tag{39}$$

Now it is clear why there is no contradiction between the fact that the probability distribution of the pointer variables of the measuring devices are identical, while the joint weak values of the product are different. It is not enough to look at the pointer variables, we have to look at their conjugate momenta as well (a different type of measurement). It is explicit in formula (39). (In formula (38) we have the complex local weak variables and in order to see their imaginary parts we need to observe the conjugated momenta of the pointer variables.)

An error analysis of this method for the example above shows that a set of measurements resulting in a deviation of $< 1\%$ and a statistical error of $10\%$ requires an ensemble of about $2 \times 10^6$ such pre and post selected systems. We obtain the weak value of the product from local
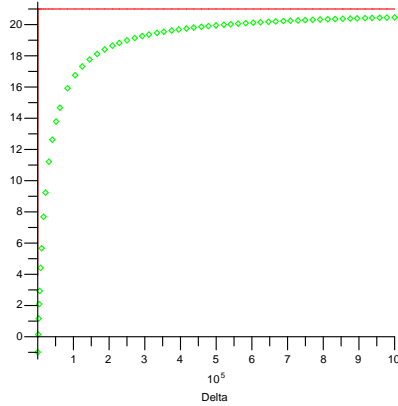


Figure 2. Comparison of local and nonlocal (unphysical) measuring devices for the measurement of a product. Expectation values of the pointer variable as a function of $\Delta$. Deviations of $1\%$ are at values of $\Delta > 100$ for the nonlocal measuring device (full line) and $\Delta > 5 \times 10^6$ for the local one (dots) .

weak measurements but at the expense of extreme errors and with a requirement of two different readings (one of Q and one of P).

For a comparison of this method and a non-local method with the (unphysical) coupling term $g(t)\sigma_z^A \sigma_z^B P$ we look at the same pre and post-selected state as before (26 ). The weak value for the product should be

$$A_w \equiv \frac{\langle \Phi | \sigma_z^A \sigma_Z^B | \Psi \rangle}{\langle \Phi | \Psi \rangle} = \frac{2 - \epsilon^2}{2 + \epsilon^2} = 21 \tag{40}$$

As can be seen in fig 2, the non local method converges much faster then the local one. A deviation of $1\%$ with an uncertainty of $10\%$ would require an ensemble of about $2 \times 10^3$ for the non-local method and about $10^{12}$ for the local method described above. It is not surprising that this method is even less practical then the one for the measurement of a sum.

## 6. Conclusions

Like eigenvalues and expectation values, weak values of a non local system can also be obtained using local methods. Such methods require larger resources, and have no pointer pointing at the desired result. Calculations using results from different types of measurement are required to arrive at the final result. We showed that the weak value of some non local variables (a sum of two or more local observables) can be measured directly. We have not found a direct way for weak measurement of a modular sum of nonlocal variables in spite of the existence of the method for strong measurement of nonlocal modular sum.

If we try to give an interpretation of weak values as *elements of reality* [22], one of the strengths of weak measurements is that it corresponds to a shift of the pointer variable by the weak value. This is not the case when making local measurements for calculating non-local values. Such methods require us to look at different pointers and use a formula for reaching the desired result. Lundeen and Steinberg [23] measured non local weak values using the method of joint weak values to measure non local weak values in an optical experiment. Their results had large deviations (more than 25%). The question of interpretation still remains open but unlike local weak values which have been measured precisely in the lab, some non-local weak values might still be thought of as accounting artifacts rather then physical observables.

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